JOURNAL OF COMPUTATIONAL PHYSICS 38, 251–256 (1980)

Note

Application of Hermite Approximation to a Boundary Value Problem

A fifth degree Hermite approximant is used to provide initial values, with error bounds, for solving the boundary value problem $y'' = f(x, y)$ by a shooting method.

1. Introduction

We consider the second order boundary value problem

$$y''(x) = f(x, y), \qquad y(0) = \alpha, \qquad y(1) = \beta, \tag{1}$$

whose solution is required on $[0, 1]$. Clearly a similar problem with boundary conditions at $x = a$ and b can be reduced to the form (1) by means of a linear change of variable. We will assume that, for all $x \in [0, 1]$ and all y, $f_y(x, y)$ is continuous and

$$f_y(x, y) \geqslant K > -\pi^2. \tag{2}$$

It then follows from a more general result due to Bailey *et al.* [1] that (1) has a unique solution. (See also Bailey *et al.* [2, p. 96].)

We propose a Hermite type of approximation for estimating $y'(0)$, which will provide initial values for solving (1) by a "shooting" method. For an account of shooting methods, see Fox [3] or Keller [4]. In this connection, note the remark by Keller in the preface to his text: "Initial-value methods are seldom advocated in the literature, but we find them extremely practical and theoretically powerful."

2. An Interpolation Process

Consider the polynomial

$$\begin{aligned} p(x) = {} & (1-x)^3\,(1+3x+6x^2)\,y_0 + x^3(10-15x+6x^2)\,y_1 \\ & + x(1-x)^3\,(1+3x)\,y_0' - x^3(1-x)(4-3x)\,y_1' \\ & + \tfrac{1}{2}x^2(1-x)^3\,y_0'' + \tfrac{1}{2}x^3(1-x)^2\,y_1'', \end{aligned} \tag{3}$$

0021-9991/80/140251–06\$02.00/0

where y_i, y_i' and y_i'' $(i=0, 1)$ denote the values of y and its first two derivatives at $x=0$ and 1. One can easily verify that this Hermite-type approximant satisfies

$$p(x)=y(x), \qquad p'(x)=y'(x), \qquad p''(x)=y''(x)$$

at $x=0$ and 1. Further, it is well known (see Davis [5]) that if $y \in C^6[0, 1]$, then given any fixed $x \in [0, 1]$ there is a number $\xi_x \in (0, 1)$ such that

$$y(x)-p(x)=x^3(x-1)^3\, y^{(6)}(\xi_x)/6!. \tag{4}$$

Given the differential equation (1), we estimate y_0' and y_1' as follows. We differentiate (4) three times and set $x=0$ and $x=1$ in turn to give

$$\begin{aligned} y^{(3)}(0) &= -60y_0+60y_1-36y_0'-24y_1' \\ &\quad -9y_0''+3y_1''-y^{(6)}(\xi_0)/5!, \end{aligned} \tag{5}$$

$$\begin{aligned} y^{(3)}(1) &= -60y_0+60y_1-24y_0'-36y_1' \\ &\quad -3y_0''+9y_1''+y^{(6)}(\xi_1)/5!. \end{aligned} \tag{6}$$

From (1) we replace y'' by $f(x, y)$ and put

$$y^{(3)}(x)=f_x(x, y(x))+f_y(x, y(x))\cdot y'(x).$$

Hence Eqs. (5) and (6) are linear in y_0' and y_1' of the form

$$\mathbf{Au}=\mathbf{b}, \tag{7}$$

where $\mathbf{u}^{\mathrm{T}}=(y_0', y_1')$ and

$$\mathbf{A}=\begin{bmatrix} 36+f_y(0, y_0) & 24 \\ 24 & 36+f_y(1, y_1) \end{bmatrix}.$$

Since the final terms on the right sides of (5) and (6) are generally unknown, we cannot solve the linear equations (7); instead, we solve the equations

$$\mathbf{A}(\mathbf{u}+\boldsymbol{\delta}\mathbf{u})=\mathbf{b}+\boldsymbol{\delta}\mathbf{b}, \tag{8}$$

where

$$\boldsymbol{\delta}\mathbf{b}=\frac{1}{5!}\begin{bmatrix} y^{(6)}(\xi_0) \\ -y^{(6)}(\xi_1) \end{bmatrix}. \tag{9}$$

From (7) and (8) it follows that the error $\boldsymbol{\delta}\mathbf{u}$ in the vector $(y_0', y_1')^{\mathrm{T}}$ satisfies the inequality

$$\|\boldsymbol{\delta}\mathbf{u}\|_\infty \leqslant \|\mathbf{A}^{-1}\|_\infty \cdot \|\boldsymbol{\delta}\mathbf{b}\|_\infty. \tag{10}$$

In order to estimate the right side of (10), we prove the following lemma.

LEMMA 1. *If*

$$\mathbf{A} = \begin{bmatrix} p+\lambda & q \\ q & p+\mu \end{bmatrix},$$

where $p > q \geqslant 0$ *and* $\lambda, \mu \geqslant K > -p+q$, *then*

$$\|\mathbf{A}^{-1}\|_\infty \leqslant 1/(p-q+K).$$

Proof. We have

$$\mathbf{A}^{-1} = ((p+\lambda)(p+\mu)-q^2)^{-1} \begin{bmatrix} p+\mu & -q \\ -q & p+\lambda \end{bmatrix}$$

and thus

$$\|\mathbf{A}^{-1}\|_\infty = ((p+\lambda)(p+\mu)-q^2)^{-1} \cdot \max[p+q+\lambda, p+q+\mu].$$

It suffices to consider the case $\lambda \geqslant \mu \geqslant K$, and we see that

$$\|\mathbf{A}^{-1}\|_\infty \leqslant \max_{\lambda \geqslant K} \frac{p+q+\lambda}{(p+\lambda)(p+K)-q^2}. \tag{11}$$

Since $(p+\lambda)(p+K)-q^2 > 0$ for $\lambda \geqslant K > -p+q$, the lemma follows immediately from the monotonicity of the function of λ on the right side of (11).

We now state:

LEMMA 2. *If* $-\pi^2 < K \leqslant f_y(x, y)$ *for* $x \in [0, 1]$ *and* $|y^{(6)}(x)| \leqslant M$ *on* $[0, 1]$, *then the boundary value problem* (1) *has a unique solution and the error in the vector* $\mathbf{u}^T = (y_0', y_1')$ *satisfies*

$$\|\boldsymbol{\delta}\mathbf{u}\|_\infty \leqslant \frac{M}{120(12+K)}. \tag{12}$$

The proof follows immediately from (2), (9), (10) and Lemma 1.

We now write

$$\mathbf{u} + \boldsymbol{\delta}\mathbf{u} = (\tilde{y}_0', \tilde{y}_1')^T$$

and let $\tilde{p}(x)$ denote the polynomial obtained when we replace y_0' and y_1' in (3) by $\tilde{y}_0'$ and $\tilde{y}_1'$, respectively. Thus $\tilde{p}$ is the polynomial which we actually construct. To compare $\tilde{p}(x)$ with $y(x)$ we write

$$y(x) - \tilde{p}(x) = (y(x) - p(x)) + (p(x) - \tilde{p}(x)). \tag{13}$$

The second term on the right of (13) may be estimated from (3):

$$|p(x) - \tilde{p}(x)| \leqslant x(1-2x^2+x^3) \cdot \|\boldsymbol{\delta}\mathbf{u}\|_\infty,$$

so that for $0 < h < 1$

$$|p(h) - \tilde{p}(h)| < h \, \|\boldsymbol{\delta}\mathbf{u}\|_\infty. \tag{14}$$

The first term on the right of (13) is estimated from (4):

$$|y(h) - p(h)| < Mh^3/6!. \tag{15}$$

Combining (14) and (15), and using (12), we deduce from (13) that

$$|y(h) - \tilde{p}(h)| \leqslant \frac{Mh}{1440}\left(\left(1 + \frac{K}{12}\right)^{-1} + 2h^2\right). \tag{16}$$

We note that the same inequality holds for $|y(1-h) - \tilde{p}(1-h)|$.

An alternative to the interpolation process which we have discussed here is given by Lanczos [6]; however, Lanczos does not provide an error estimate for his method.

3. Numerical Example

We will illustrate our method by obtaining a numerical solution of the test problem

$$y'' = \sin y, \qquad y(0) = 0, \qquad y(1) = 1. \tag{17}$$

First we set up the linear equations (5) and (6), omitting the terms involving $y^{(6)}$, and solve to give

$$y'(0) \simeq 0.854266, \qquad y'(1) \simeq 1.288190.$$

We choose $h = 0.1$ and evaluate $\tilde{p}(0.1)$; that is, using (3) with the above approximate values for y_0' and y_1'. This yields

$$y(0.1) \simeq 0.085570.$$

The most obvious discretization of (1) at $x = x_n$ is

$$y_{n+1} - 2y_n + y_{n-1} = h^2 f(x_n, y_n), \tag{18}$$

which has a truncation error of $O(h^2)$. However, with starting values $y_0 = 0$, $y_1 = 0.085570$, we use the $O(h^4)$ method

$$y_{n+1} - 2y_n + y_{n-1} = (1/12)\, h^2 (f(x_{n+1}, y_{n+1}) + 10 f(x_n, y_n) + f(x_{n-1}, y_{n-1})). \tag{19}$$

Since (19) is implicit, (18) may be used at each stage as a predictor for (19). The resulting solution is displayed in the column y_n of Table I. This gives $y_{10} = 0.997382$, which differs by 0.002618 from the given boundary value $y(1) = 1$.

We use Fox's η-method (see [3]) to compute an improved numerical solution y_n^*,

TABLE I

Numerical Solution of $y'' = \sin y$, $y(0) = 0$, $y(1) = 1$

n	y_n	η_n	y_n^*	$\tilde{p}(x_n)$	$y(x_n)$
0	0	0	0	0	0
1	0.085570	0.000262	0.085798	0.085570	0.085797
2	0.171995	0.000527	0.172453	0.172006	0.172452
3	0.260132	0.000796	0.260824	0.260189	0.260823
4	0.350841	0.001073	0.351774	0.351006	0.351773
5	0.444987	0.001360	0.446169	0.445347	0.446169
6	0.543438	0.001659	0.544880	0.544092	0.544879
7	0.647058	0.001973	0.648773	0.648105	0.648773
8	0.756705	0.002303	0.758707	0.758228	0.758706
9	0.873214	0.002649	0.875517	0.875270	0.875516
10	0.997382	0.003011	1.000000	1.000000	1.000000

which appears in the fourth column of our table. For completeness, we now describe how this correction is performed. Let us denote by $z(x)$ the function which satisfies the given differential equation together with boundary conditions $z(0) = 0$, $z(1) = y_{10}$. Now write $y(x) = z(x) + \eta(x)$, where y is the exact solution of (17). Thus

$$(z + \eta)'' = f(x, z + \eta)$$

and, on linearizing, we obtain

$$\eta'' \simeq \eta \cdot f_y(x, z). \tag{20}$$

We assume that (20) holds exactly and seek the solution which satisfies the boundary conditions $\eta(0) = 0$, $\eta(1) = 0.002618$. We choose $\eta_0 = 0$, $\eta_1 = 0.000262$ (obtained by assuming that η is itself linear; this choice of η_1 is not critical). We again use method (19) and the resulting solution for the η_n is given in the table. Note that this gives $\eta_{10} = 0.003011$ instead of the required value of 0.002618. Due to the linearity of (20) we obtain the solution which matches the correct boundary conditions on multiplying the η_n in the table by (0.002618)/(0.003011). Finally, the solution to (17) is obtained by adding the scaled η_n to the y_n, to give y_n^* in the fourth column of the table. This may be compared with the Hermite interpolating polynomial $\tilde{p}(x)$ and the exact solution $y(x)$, which are given in the last two columns of the table.

4. Concluding Remarks

Our error estimate (12) shows that the error in the approximation to $y'(0)$ depends on the size of $y^{(6)}(x)$. If $y^{(6)}(x)$ is large compared to $y'(x)$, the approximation may be poor, despite the helpful large denominator on the right of (12).

Nonetheless, our technique seems to be generally worth using, especially as it requires very little additional computation. Indeed, if $y^{(6)}(x)$ is not large, $\tilde{p}(x)$ may for some purposes serve as an adequate approximation to the solution y on the whole interval $[0\ 1]$.

Finally, although we have confined the presentation of our technique to the equation $y'' = f(x, y)$, we may also apply it to the equation $y'' = f(x, y, y')$. In the latter case the two equations in $y'(0)$ and $y'(1)$ are nonlinear.

Acknowledgments

The authors wish to thank the referees for their most helpful comments.

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Received: June 18, 1979; revised: November 29, 1979

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